Malaysian Journal of Mathematical Sciences 17(4): 731–754 (2023)
https://doi.org/10.47836/mjms.17.4.13

Malaysian Journal of Mathematical Sciences

Journal homepage: https://mjms.upm.edu.my

The Effect of Alternative Resource and Refuge on the Dynamical Behavior of Food Chain Model

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Received: 3 October 2023
Accepted: 4 December 2023

Abstract

This article examines and proposes a dietary chain model with a prey shelter and alternative food sources. It is anticipated that mid-predators' availability is positively correlated with the number of refuges. The solution's existence and exclusivity are examined. It is established that the solution is bounded. It is explored whether all potential equilibrium points exist and are locally stable. The Lyapunov approach is used to investigate the equilibrium points' worldwide stability. Utilizing a Sotomayor theorem application, local bifurcation is studied. Numerical simulation is used to better comprehend the dynamics of the model and define the control set of parameters.

Keywords: food chain; refuge; alternative food; stability; Lyapunov method; bifurcation.

1 Introduction

Numerous studies have demonstrated a rise in interest in the use of mathematical models in ecology, for examples, the Allee effect's influence on the dynamics of a delayed prey-predator model with a Hattaf-Yousfi functional response was investigated by Bouziane et al. [3]. By adjusting the superpredator's kill rate parameter, Das and Bhardwaj [6] have studied the intricacy of temporal dynamics in the three-species food web model, including the Holling type II functional response. Holling [12] covered the most recent developments in a systems study of predation, which addresses the functional reactions of predators to the density of their prey. The rivalry between two prey and one predator, taking into account both a Holling type II functional response and an additive Allee effect in the predator population, was examined by Kumar and Gunasundari [20]. In [17], a prey-predator paradigm with independent harvesting in both species and prey refuge is discussed. Mondal and Samanta [22], who included nonlinear prey refuge to prevent predator extinction, examined the fear effect's outcomes for the dynamics of predator-prey interaction. An analysis using mathematics has been conducted on prey-predator ecological models in which the predator has an alternate food source and the prey has a partial cover [28]. However, Tian and Xu [37] examined a predator-prey system with Holling type II functional response and stage structure. It has garnered considerable interest from numerous scientific fields. Early studies revealed that the availability of resources is crucial. Numerous studies looked at how alternate resources affected food chains. Researchers found that when resources were rare, populations would decline as individuals battled for access to the scarce resources, according to Senthamara and Vijayalakhmi [35].

Numerous studies by scientists have examined how the movements of societies and populations are impacted by births and deaths, for example, a mathematical model consisting of two prey and one predator with a Beddington-DeAngelis functional response is proposed and analyzed by Naji and Balasim [25]. However, Naji et al. [27] have proposed and studied a model food chain involving a specialist and a generalist predator. Therefore, the food chain is crucial since all living things, regardless of size, depend on one another to survive. Academic journals have given a lot of attention to the study of dietary chains and how animals stay alive by consuming other species. It educates people that every living thing is reliant on other living things for survival. It is crucial to realize that even a small disruption in the natural food chain or feeding habits can cause a large number of species to change their behavior.Jabr and Bahlool [14] have studied the role of a prey refuge, depending on both species, in the dynamics of a food web system. Satar and Naji [34] investigated the stability and bifurcation of a prey-predator-scavenger model in the presence of toxicants and harvesting. Ws et al. [32] proposed and studied a mathematical model of three-species food chain interaction with mixed functional response.

On the other hand, a refuge, a word from ecology that refers to a situation in which an organism gains safety by hiding in obscure locations, piqued the attention of many scientists. A diseased prey-predator model with prey serving as a haven and predators providing food was presented and examined by Abdulghafour and Naji [1]. On the other hand, Bahlool et al. [2] examined chaos and order in a prey-predator paradigm that included sickness, refuge, and harvesting. Prey refuge was added by Gkana and Zachilas [11] to a prey-predator model that included population breakouts and a Holling type I functional response. Huang et al. [13] looked into the stability analysis of a prey-predator model using a Holling type III response function that included a prey refuge. Kar [16] examined the stability study of a predator-prey model with a prey refuge. Mondal and Samanta [24] examined the dynamics of a predator-prey system that provides additional food, where the prey's refuge is reliant on both species and there is a constant harvest for the predators. Mondal et al. [23] examined the dynamics of a predator-prey population under nonlinear prey refuge and fear effects in the presence of resource subsidy. Sarwardi et al. [33]

examined the dynamic behavior of a two-predator model with prey shelter.

Additionally, De Rossi et al. [8] examined the impact of refuge on prey and the importance of the refuge component. Refuge in prey was hypothesized and researched by Naji and Majeed [26] as a protective characteristic versus predation and harvesting obtained from predators. Sih [36] investigated the effects of prey refuge and hunting activities. After Das et al. [7] examined the effects of refuges' availability on the interaction of predators and prey, Molla et al. [21]proposed a mathematical framework for prey-predator enabling prey refuge dependent on each prey and predator species. According to Ko and Ryu's findings in [18], the dynamics of the system integrating a prey refuge with homogenous Neumann boundary conditions are significantly influenced by the configuration of the functional response. The unstable impact of the prey refuge under specific circumstances is particularly the intriguing conclusion. A prey-predator model based on the second kind of Holling predation rate that takes into account the harvesting process of each species was described by Kar et al. [15]. They have also thought about collecting each species and delaying the reproductive cycle of the predator group.

The availability of substitute foods can influence biological control through a number of methods, according to numerous studies. With a third kind of Holling predation rate, Agarwal and Kumar [19] investigated the impact of substitute resources for top predators in the dietary chain model. According to Sahoo's theory [30], substitute nourishment has a stabilizing effect on predator-prey interactions. In later works, different investigations on the effects of an alternate meal can be found. See for examples: a prey-predator model with a haven for prey and extra food for predators in a changing environment was presented and researched by Das and Samanta [4]. In contrast, Das et al.'s study [5] examined how different diets might regulate chaotic dynamics in a predator-prey scenario when the predator had a sickness. In a time-varying prey-predator model with different delays and substitute food sources for predators, Devi and Jana [9] investigated the function of fear. Ghosh et al. [10] examined prey-predator dynamics, wherein the predator receives more food from the prey shelter. Sahoo et al. [31] examined the impact of substitute resources on the dynamics of the harvested-predator-prey model. Given the aforementioned information, in order to address the consequences of alternative resources for a top predator, the researchers were driven to examine the mathematical depiction of prey refuge in a three-species dietary chain using a second kind of Holling predation rate.

2 Formulation of the Mathematical Framework

According to the following hypotheses, a mathematical framework of a three-species dietary chain that consists of prey, mid-predators, and top predators and combines a prey refuge and a substitute source of food for a top predator is developed:

1. The model is designed to include three species, with $x(t)$ representing the prey populations at time t and $y(t)$ and $z(t)$ representing the mid- and top predator populations at time t, respectively. The killing processes have been carried out in accordance with the second kind of Holling predation rate.
2. While the prey expands logistically in the absence of a predator, the mid-predator expands logistically in addition to the food obtained by its predation on the prey. Lack of food causes the mid-predator to drop rapidly.
3. The upper predator also has other food sources from its surroundings in addition to the food it takes from the mid-predator. The upper predators engage in intra-specific conflict with

one another. In addition, when nourishment is scarce, the top predator drops rapidly.

4. Last, but not least, it is hypothesized that the remaining prey population is vulnerable to being devoured by a mid-predator because the prey species defends itself by offering a refuge that is proportional to the number of mid-predators.

The next collection of nonlinear differential equations of first order may be utilized for expressing the evolution of the food chain under the aforementioned suppositions:

$$\begin{aligned}\frac{dx}{dt} &= rx\left(1-\frac{x}{K}\right)-\frac{a_1\left(x-x_R\right)y}{a+\left(x-x_R\right)},\\ \frac{dy}{dt} &= sy\left(1-\frac{y}{L}\right)+\frac{a_2\left(x-x_R\right)y}{a+\left(x-x_R\right)}-\frac{b_1yz}{b+y}-d_1y,\\ \frac{dz}{dt} &= \frac{b_2yz}{b+y}+A_1z-A_2z^2-d_2z,\end{aligned} \tag{1}$$

where Table 1 describes every parameter, and they're all thought of as positive constants.
If $x_R = cxy$ is substituted, system (1) will be as follows:

$$\begin{aligned}\frac{dx}{dt} &= x\left[r\left(1-\frac{x}{K}\right)-\frac{a_1\left(1-cy\right)y}{a+x\left(1-cy\right)}\right]=xF_1,\\ \frac{dy}{dt} &= y\left[s\left(1-\frac{y}{L}\right)+\frac{a_2\left(1-cy\right)x}{a+x\left(1-cy\right)}-\frac{b_1z}{b+y}-d_1\right]=yF_2,\\ \frac{dz}{dt} &= z\left[\frac{b_2y}{b+y}+A_1-A_2z-d_2\right]=zF_3,\end{aligned} \tag{2}$$

with $x\left(0\right)\geq 0$, $y(0)\geq 0$ and $z(0)\geq 0$. The functions of the vector $F=\left(F_1,F_2,F_3\right)^T$ are Lipschitzian because they are clearly continuous and have continuously differential functions on $\Xi=\left\{(x,y,z)\in\mathbb{R}^3 : x\geq 0,\ y\geq 0,\ z\geq 0\right\}$. Consequently, the solution to system (2) already exists and is unique.

Table 1: The parameters description.

Parameter	Description
a_1 and b_1	The catching rates of y and z respectively.
a and b	The half-saturation constants for y and z respectively.
a_2 and b_2	The food transformation rates into y and z respectively.
r and K	The intrinsic growth rate and carrying capacity for x.
s and L	The real growth rate and carrying capacity for the y species.
d_1	The natural mortality rate of the y species.
A_1	A substitute food rate for the z species.
A_2	The intraspecific competition rate within the population of the z species.
d_2	The natural mortality rate of the z species.
$c\in[0,\ 1]$	The proportionality constant of prey refuge with y, so that $y\leq\frac{1}{c}$.

3 Limitations of the System

This section investigates the uniform boundedness of the system (2)'s trajectory as stated in the following theorem.

Theorem 3.1. *System* (2) *is a dissipative system in* Ξ.

Proof. It is sufficient to demonstrate that all the trajectories of system (2) are uniformly bounded in order to demonstrate this theorem. If you take a look at the function $w(t)$, which is the sum of solutions in Ξ, as represented by $w(t) = x(t) + y(t) + z(t)$, then,

$$\begin{aligned}\frac{dw}{dt} &= \frac{dx}{dt} + \frac{dy}{dt} + \frac{dz}{dt} \\ &= rx\left(1 - \frac{x}{K}\right) - \frac{(a_1 - a_2)(x - x_R)\,y}{a + (x - x_R)} + sy\left(1 - \frac{y}{L}\right) \\ &\quad - \frac{(b_1 - b_2)\,yz}{b + y} - d_1 y + A_1 z\left(1 - \frac{z}{\left(\frac{A_1}{A_2}\right)}\right) - d_2 z,\end{aligned}$$

$$\frac{dw}{dt} \le \frac{(r+1)^2 K}{4r} + \frac{sL}{4} + \frac{{A_1}^2}{4A_2} - \mu\,(x + y + z) = \rho - \mu w,$$

where $\mu = \min\{1, d_1, d_2\}$ and $\rho = \dfrac{(r+1)^2 K}{4r} + \dfrac{sL}{4} + \dfrac{{A_1}^2}{4A_2}$, therefore we obtain,

$$\frac{dw}{dt} + \mu w \le \rho.$$

Since the ensuing differential inequality can be solved, it is discovered that for $t \to \infty$; $w(t) \le \dfrac{\rho}{\mu}$, which results in the dissipative of the system (2). ☐

According to the aforementioned theorem, system (2) has an attractor that belongs to the Ξ category.

4 The Equilibria of the System

For system (2), there are a maximum of eight equilibrium points (EPs). Here are some descriptions of them:

1. The points $\mathbf{e}_0 = (0,0,0)$ and $\mathbf{e}_1 = (\breve{x},0,0) = (K,0,0)$ located unconditionally.
2. The point $\mathbf{e}_2 = (0,\hat{y},0) = \left(0, \dfrac{(s-d_1)\,L}{s}, 0\right)$ is available if the following requirement occurs,

$$s > d_1. \tag{3}$$

3. The point $\mathrm{e}_3 = (0, 0, \overline{z}) = \left(0, 0, \dfrac{A_1 - d_2}{A_2}\right)$ is available if the following requirement occurs,

$$A_1 > d_2. \tag{4}$$

4. The y free point $\mathrm{e}_4 = (\breve{x}, 0, \overline{z}) = \left(K, 0, \dfrac{A_1 - d_2}{A_2}\right)$ is available if the requirement (4) occurs.

5. The x free point $\mathrm{e}_5 = (0, \widetilde{\widetilde{y}},\ \widetilde{\widetilde{z}})$ is available if the following requirements occur,

$$d_2 < \frac{b_2 y + A_2(b+y)}{b+y}, \tag{5}$$

$$\left.\begin{array}{c} \mathcal{B}_3 > 0 \text{ and } \mathcal{B}_1 < 0, \\ or \\ \mathcal{B}_3 > 0 \text{ and } \mathcal{B}_2 > 0, \end{array}\right\}, \tag{6}$$

where $\widetilde{\widetilde{z}} = \dfrac{b_2 y + (A_1 - d_2)(b+y)}{A_2(b+y)}$, while the third-degree polynomial that is provided below has $\widetilde{\widetilde{y}}$ as a positive root,

$$\mathcal{B}_0 y^3 + \mathcal{B}_1 y^2 + \mathcal{B}_2 y + \mathcal{B}_3 = 0,$$

where,

$$\begin{aligned}
\mathcal{B}_0 &= -sA_2 < 0,\\
\mathcal{B}_1 &= (s - d_1)A_2 L - 2sbA_2,\\
\mathcal{B}_2 &= 2\left(s - d_1\right) A_2 L b - sA_2 b^2 - b_1 b_2 L - L b_1 A_1 + L b_1 d_2,\\
\mathcal{B}_3 &= \left(s - d_1\right) A_2 L b^2 - b_1 A_1 b L + L b b_1 d_2.
\end{aligned}$$

6. The z free point $\mathrm{e}_6 = (\check{\check{x}}, \check{\check{y}},\ 0)$ is available if the following requirements occur,

$$s\left(1 - \frac{\check{\check{y}}}{L}\right) < d_1 < a_2 + s\left(1 - \frac{\check{\check{y}}}{L}\right), \tag{7}$$

with a single set from the list below,

$$\left.\begin{array}{lllll}
\mathbb{Q}_0 > 0, & \mathbb{Q}_1 > 0, & \mathbb{Q}_2 > 0, & \mathbb{Q}_3 > 0, & \mathbb{Q}_4 < 0,\\
\mathbb{Q}_0 > 0, & \mathbb{Q}_1 > 0, & \mathbb{Q}_2 > 0, & \mathbb{Q}_3 > 0, & \mathbb{Q}_4 > 0,\\
\mathbb{Q}_0 > 0, & \mathbb{Q}_1 > 0, & \mathbb{Q}_2 > 0, & \mathbb{Q}_3 < 0, & \mathbb{Q}_4 < 0,\\
\mathbb{Q}_0 > 0, & \mathbb{Q}_1 > 0, & \mathbb{Q}_2 < 0, & \mathbb{Q}_3 < 0, & \mathbb{Q}_4 < 0,\\
\mathbb{Q}_0 > 0, & \mathbb{Q}_1 < 0, & \mathbb{Q}_2 < 0, & \mathbb{Q}_3 < 0, & \mathbb{Q}_4 < 0,
\end{array}\right\}, \tag{8}$$

where $\check{\check{x}} = \dfrac{a\left[d_1 - s\left(1 - \frac{\check{\check{y}}}{L}\right)\right]}{\left[a_2 - d_1 + s\left(1 - \frac{\check{\check{y}}}{L}\right)\right]\left(1 - c\check{\check{y}}\right)}$, while the fifth-degree polynomial that is provided below has $\check{\check{y}}$ as a positive root,

$$\mathbb{Q}_5 y^5 + \mathbb{Q}_4 y^4 + \mathbb{Q}_3 y^3 + \mathbb{Q}_2 y^2 + \mathbb{Q}_1 y + \mathbb{Q}_0 = 0,$$

with,

$$\mathbb{Q}_0 = aa_2r(a_2 - d_1 + s) - \frac{a^2a_2r}{K}(d_1 - s),$$
$$\mathbb{Q}_1 = aa_2{}^2cr + aa_2cd_1r - 2a_1a_2d_1 - 2a_1a_2s - a_1d_1{}^2 + 2a_1d_1s - a_1s^2 + aa_2crs + acd_1{}^2r - 2acd_1rs + acrs^2 - \frac{aa_2rs}{L} - \frac{a^2a_2rs}{KL} + 2a_1a_2d_1,$$
$$\mathbb{Q}_2 = \frac{aa_2crs}{L} + \frac{2a_1a_2s}{L} - \frac{2a_1d_1s}{L} + \frac{2a_1s^2}{L} + 2a_1a_2{}^2c + 4a_1a_2cs + 2a_1cd_1{}^2 - 4a_1cd_1s + 2a_1cs^2 - 4a_1a_2cd_1,$$
$$\mathbb{Q}_3 = \frac{-a_1s^2}{L^2} - \frac{4a_1a_2cs}{L} + \frac{4a_1cd_1s}{L} + \frac{2a_1cs^2}{L} - a_1a_2{}^2c^2 + 2a_1a_2c^2d_1 - 2a_1a_2c^2s^2 - a_1c^2d_1{}^2 + 2a_1c^2d_1s - a_1c^2s^2,$$
$$\mathbb{Q}_4 = \frac{2a_1cs^2}{L^2} + \frac{2a_1c^2s}{L}(a_2 - d_1 + s),$$
$$\mathbb{Q}_5 = -\frac{c^2a_1s^2}{L^2}.$$

7. The survival point $\mathbf{e}_7 = (x^*, y^*, z^*)$, where $z^* = \dfrac{b_2y + (A_1 - d_2)(b + y)}{A_2(b + y)}$, while (x^*, y^*) represents an intersection point of the next two isoclines,

$$\left.\begin{aligned} g_1(x,y) &= aKr + Krx - cKrxy - arx - rx^2 + crx^2y - a_1Ky + a_1cKy^2 = 0,\\ g_2(x,y) &= s - \frac{sy}{L} + \frac{a_2x(1-cy)}{a + x(1-cy)} - \frac{b_1}{b+y}\left(\frac{b_2y + (A_1 - d_2)(b+y)}{A_2(b+y)}\right) - d_1 = 0, \end{aligned}\right\}.$$

Direct computation shows that $\mathbf{e}_7$ is available uniquely if the following requirements occur,

$$\left.\begin{aligned} & bA_2s + b_1d_2 < b_1A_1 + bA_2d_1 < bA_2(s + a_2) + b_1d_2,\\ & h_1 < h_2,\\ & \frac{dy}{dx} = -\frac{(\partial g_1/\partial x)}{(\partial g_1/\partial y)} > 0,\\ & \frac{dy}{dx} = -\frac{(\partial g_2/\partial x)}{(\partial g_2/\partial y)} < 0, \end{aligned}\right\}, \tag{9}$$

with $h_1 = -\dfrac{(a-K)}{2} + \dfrac{1}{2}\sqrt{(a-K)^2 + 4aK}$, and $h_2 = \dfrac{baA_2(d_1 - s) + b_1a(A_1 - d_2)}{bA_2(s + a_2) - b_1(A_1 - d_2) - bA_2d_1}$.

5 Stability and Bifurcation Locally

In this part, using the linearization approach and the Sotomayor theorem [29], respectively, we explore the local stability and bifurcation of all potential EPs of the system (2).

It is simple to confirm that $\lambda_{01} = r > 0$, $\lambda_{02} = s - d_1$ and $\lambda_{03} = A_1 - d_2$ are the values of the roots of the variational matrix (VM) at $\mathbf{e}_0 = (0, 0, 0)$. So, $\mathbf{e}_0$ is an unstable point because positive eigenvalues occur.

Theorem 5.1. *The EP,* $\mathbf{e}_1 = (\check{x}, 0, 0)$*, is asymptotically stable locally if the following requirements occur,*

$$s + \frac{a_2 K}{a + K} < d_1, \tag{10}$$

$$A_1 < d_2. \tag{11}$$

However, there is a transcritical bifurcation (TCB) a round $\mathbf{e}_1$ *when the* $A_1 = d_2 (\equiv A_1^*)$*.*

Proof. Since the VM at the point $\mathbf{e}_1$ can be written as,

$$J(\mathbf{e}_1) = \begin{bmatrix} -r & -\dfrac{a a_1 K}{(a+K)^2} & 0 \\ 0 & s + \dfrac{a_2 K}{a+K} - d_1 & 0 \\ 0 & 0 & A_1 - d_2 \end{bmatrix}.$$

Obviously, $J(\mathbf{e}_1)$ has the eigenvalues $\lambda_{11} = -r < 0$, $\lambda_{12} = s + \dfrac{a_2 K}{a+K} - d_1$ and $\lambda_{13} = A_1 - d_2$, which are negative if the conditions (10) and (11) holds. Thus, $\mathbf{e}_1$ is asymptotically stable locally.

Now, when $A_1 = A_1^*$, then the third eigenvalue of $J(\mathbf{e}_1, A_1^*)$ becomes $\check{\lambda}_{13} = 0$. Moreover, it is obtained that $\check{V} = (0,0,1)^T$ is the eigenvector related to $\check{\lambda}_{13} = 0$ of $J(\mathbf{e}_1, A_1^*)$, and $\check{\varphi} = (0,0,1)^T$ is the eigenvector related to $\check{\lambda}_{13} = 0$ of $[J(\mathbf{e}_1, A_1^*)]^T$. Also,

$$\check{\varphi}^T \left[\frac{df}{dA_1}(\mathbf{e}_1, A_1^*) \right] = 0, \qquad \text{where } f = (xF_1, xF_2, xF_3)^T,$$

$$\check{\varphi}^T \left[\frac{d}{dX} f_{A_1}(\mathbf{e}_1, A_1^*) \check{V} \right] = 1 \neq 0, \qquad \text{where } X = (x, y, z)^T,$$

$$\check{\varphi}^T \left[\frac{d^2}{dX^2} f(\mathbf{e}_1, A_1^*).(\check{V}, \check{V}) \right] = -2A_2 \neq 0.$$

Thus, by Sotomayor's theorem, the TCB occurs. □

Theorem 5.2. *The EP,* $\mathbf{e}_2 = (0, \hat{y}, 0) = \left(0, \dfrac{(s-d_1)L}{s}, 0\right)$*, is asymptotically stable locally if the following requirements occur,*

$$r + \frac{a_1 c \hat{y}^2}{a} < \frac{a_1}{a}\hat{y}, \tag{12}$$

$$\frac{b_2 \hat{y}}{b + \hat{y}} + A_1 < d_2. \tag{13}$$

However, there is a TCB a round $\mathbf{e}_2$ *when the* $d_2 = \dfrac{b_2 \hat{y}}{b + \hat{y}} + A_1 (\equiv d_2^*)$*.*

Proof. From the VM at point $\mathbf{e}_2$ that can be represented as:

$$J(\mathbf{e}_2) = \begin{bmatrix} r - \dfrac{a_1 \hat{y}}{a} + \dfrac{a_1 c \hat{y}^2}{a} & 0 & 0 \\ -\dfrac{a_2(1 - c\hat{y})\hat{y}}{a} & d_1 - s & -\dfrac{b_1 \hat{y}}{b + \hat{y}} \\ 0 & 0 & \dfrac{b_2 \hat{y}}{b + \hat{y}} + A_1 - d_2 \end{bmatrix}.$$

It is obvious that the eigenvalues are $\lambda_{21} = r - \frac{a_1\hat{y}}{a} + \frac{a_1 c\hat{y}^2}{a}$, $\lambda_{22} = d_1 - s < 0$, and $\lambda_{23} = \frac{b_2\hat{y}}{b+\hat{y}} + A_1 - d_2$. Therefore, the EP $\mathbf{e}_2$ is asymptotically stable locally if the conditions (12) and (13) hold.

Now, when $d_2 = d_2^*$, then $\widehat{\lambda}_{23} = 0$ will be the third eigenvalue for $J(\mathbf{e}_2, d_2^*)$. Moreover, it is obtained that $\hat{V} = (0, \beta, 1)^T$, where $\beta = -\frac{\hat{b}_{23}}{\hat{b}_{22}} < 0$, is the eigenvector related to $\widehat{\lambda}_{23} = 0$ of $J(\mathbf{e}_2, d_2^*)$, and $\widehat{\varphi} = (0, 0, 1)^T$ is the eigenvector related to $\check{\lambda}_{13} = 0$ of $[J(\mathbf{e}_2, d_2^*)]^T$. Also,

$$\widehat{\varphi}^T \left[\frac{df}{dA_1}(\mathbf{e}_2, d_2^*)\right] = 0,$$

$$\widehat{\varphi}^T \left[\frac{d}{dX} f_{d_2}(\mathbf{e}_2, d_2^*)\hat{V}\right] = 1 \neq 0,$$

$$\widehat{\varphi}^T \left[\frac{d^2}{dX^2} f(\mathbf{e}_2, d_2^*).(\hat{V}, \hat{V})\right] = \left[\frac{2bb_2}{(b+y)^2}\beta - 2A_2\right] \neq 0.$$

Thus, the TCB occurs. □

It is simple to confirm that the VM at EP $\mathbf{e}_3 = (0, 0, \bar{z}) = \left(0, 0, \frac{A_1 - d_2}{A_2}\right)$ has the eigenvalues by $\lambda_{31} = r > 0$, $\lambda_{32} = s - \frac{b_1}{b}\bar{z} - d_1$ and $\lambda_{33} = -A_2\bar{z} < 0$. Therefore, $\mathbf{e}_3$ is a saddle point.

Theorem 5.3. *The y free EP, $\mathbf{e}_4 = (\breve{x}, 0, \bar{z}) = \left(K, 0, \frac{A_1 - d_2}{A_2}\right)$, is asymptotically stable locally if the following requirement occurs,*

$$s + \frac{a_2 K}{a+K} < \frac{b_1}{b}\left(\frac{A_1 - d_2}{A_2}\right) + d_1. \tag{14}$$

However, there is a TCB a round $\mathbf{e}_4$ when $d_1 = s + \frac{a_2 K}{a+K} - \frac{b_1}{b}\left(\frac{A_1 - d_2}{A_2}\right) (\equiv d_1^*)$.

Proof. Direct computation shows that the VM at EP $\mathbf{e}_4$ is given by,

$$J(\mathbf{e}_4) = \begin{bmatrix} -r & -\frac{a a_1 K}{(a+K)^2} & 0 \\ 0 & s + \frac{a_2 K}{a+K} - \frac{b_1}{b}\left(\frac{A_1 - d_2}{A_2}\right) - d_1 & 0 \\ 0 & \frac{b_2}{b}\left(\frac{A_1 - d_2}{A_2}\right) & -(A_1 - d_2) \end{bmatrix} = [b_{ij}].$$

Consequently, the eigenvalues of $J(\mathbf{e}_4)$ are $\lambda_{41} = -r < 0$, $\lambda_{42} = s + \frac{a_2 K}{a+K} - \frac{b_1}{b}\left(\frac{A_1 - d_2}{A_2}\right) - d_1$, and $\lambda_{43} = -(A_1 - d_2) < 0$. Therefore, $\mathbf{e}_4$ is asymptotically stable locally under condition (14).

Obviously, for $d_1 = d_1^*$ the second eigenvalues becomes $\lambda_{42} = 0$. Furthermore, it is obtained that $\overline{\overline{V}} = (\alpha_1, 1, \alpha_2)^T$ with $\alpha_1 = -\frac{b_{12}}{b_{11}} < 0$ and $\alpha_2 = -\frac{b_{32}}{b_{33}} > 0$, as an eigenvector of $J(\mathbf{e}_4)$ related with zero eigenvalue. While $\overline{\overline{\varphi}} = (0, 1, 0)^T$ is an eigenvector of $[J(\mathbf{e}_4)]^T$ related to zero eigenvalue.

Also, it is easy to verify that,

$$\overline{\overline{\varphi}}^T \left[\frac{df}{dd_1}\left(\mathbf{e}_4, d_1^*\right)\right] = 0,$$

$$\overline{\overline{\varphi}}^T \left[\frac{d}{dX} f_{d_1}\left(\mathbf{e}_4, d_1^*\right) \overline{\overline{V}}\right] = 1 \neq 0,$$

$$\overline{\overline{\varphi}}^T \left[\frac{d^2}{dX^2} f\left(\mathbf{e}_4, d_1^*\right)(\overline{\overline{V}}, \overline{\overline{V}})\right] = 2\left[\frac{aa_2}{(a+K)^2}\alpha_1 - \frac{b_1}{b}\alpha_2 - \frac{s}{L} - \frac{aa_2cK}{(a+K)^2} - \frac{b_1}{b}\left(\frac{A_1-d_2}{A_2}\right)\right] \neq 0.$$

Hence, the TCB occurs and the proof is done. □

Theorem 5.4. *The x free EP, $\mathbf{e}_5 = (0, \tilde{\tilde{y}}, \tilde{\tilde{z}})$, is asymptotically stable locally if the following requirements occur,*

$$r < \frac{a_1(1-c\tilde{\tilde{y}})\tilde{\tilde{y}}}{a}, \tag{15}$$

$$\frac{b_1\tilde{\tilde{z}}}{(b+\tilde{\tilde{y}})^2} < \frac{s}{L}. \tag{16}$$

However, there is a TCB a round $\mathbf{e}_5$ when $= \dfrac{a_1\tilde{\tilde{y}}(1-c\tilde{\tilde{y}})}{a}$ $(\equiv r^*)$ provided that,

$$-\frac{a_1\tilde{\tilde{y}}(1-c\tilde{\tilde{y}})}{aK} + \frac{a_1\tilde{\tilde{y}}(1-c\tilde{\tilde{y}})^2}{a^2} - \frac{a_1\left(1-2c\tilde{\tilde{y}}\right)}{a}\rho_1 \neq 0. \tag{17}$$

Proof. The VM of system (2) at $\mathbf{e}_5$ can be determined by,

$$J(\mathbf{e}_5) = \begin{bmatrix} r - \dfrac{a_1\left(1-c\tilde{\tilde{y}}\right)\tilde{\tilde{y}}}{a} & 0 & 0 \\ \dfrac{a_2\left(1-c\tilde{\tilde{y}}\right)\tilde{\tilde{y}}}{a} & -\dfrac{s\tilde{\tilde{y}}}{L} + \dfrac{b_1\tilde{\tilde{y}}\tilde{\tilde{z}}}{\left(b+\tilde{\tilde{y}}\right)^2} & -\dfrac{b_1\tilde{\tilde{y}}}{b+\tilde{\tilde{y}}} \\ 0 & \dfrac{bb_2\tilde{\tilde{z}}}{\left(b+\tilde{\tilde{y}}\right)^2} & -A_2\tilde{\tilde{z}} \end{bmatrix} = \left[\tilde{\tilde{b}}_{ij}\right].$$

Direct computation shows that $J(\mathbf{e}_5)$ has the following eigenvalues,

$$\lambda_{51} = r - \frac{a_1(1-c\tilde{\tilde{y}})\tilde{\tilde{y}}}{a}, \qquad \lambda_{52} = \frac{T_5}{2} - \frac{1}{2}\sqrt{{T_5}^2 - 4D_5}, \qquad \lambda_{53} = \frac{T_5}{2} + \frac{1}{2}\sqrt{{T_5}^2 - 4D_5},$$

where,

$$T_5 = \tilde{\tilde{y}}\left[-\frac{s}{L} + \frac{b_1\tilde{\tilde{z}}}{\left(b+\tilde{\tilde{y}}\right)^2}\right] - A_2\tilde{\tilde{z}},$$

$$D_5 = -\tilde{\tilde{y}}\left[-\frac{s}{L} + \frac{b_1\tilde{\tilde{z}}}{\left(b+\tilde{\tilde{y}}\right)^2}\right]\left(A_2\tilde{\tilde{z}}\right) + \left[\frac{bb_2\tilde{\tilde{z}}}{\left(b+\tilde{\tilde{y}}\right)^2}\right]\left[\frac{b_1\tilde{\tilde{y}}}{b+\tilde{\tilde{y}}}\right].$$

Therefore, it is simple to check that conditions (15) and (16) guarantee that all the above eigenvalues have negative real parts. Hence, $\mathbf{e}_5$ is asymptotically stable locally. Moreover, it is obtained that $\lambda_{51} = 0$ when $= r^*$, and $\tilde{\tilde{V}} = (1, \rho_1, \rho_2)^T$, where $\rho_1 = -\frac{\tilde{\tilde{b}}_{21}\tilde{\tilde{b}}_{33}}{\tilde{\tilde{b}}_{22}\tilde{\tilde{b}}_{33} - \tilde{\tilde{b}}_{23}\tilde{\tilde{b}}_{32}} > 0$ and $\rho_2 = \frac{\tilde{\tilde{b}}_{21}\tilde{\tilde{b}}_{32}}{\tilde{\tilde{b}}_{22}\tilde{\tilde{b}}_{33} - \tilde{\tilde{b}}_{23}\tilde{\tilde{b}}_{32}} > 0$, is the eigenvector of $J(\mathbf{e}_5)$ related to $\lambda_{51} = 0$. While $\tilde{\tilde{\varphi}} = (1, 0, 0)^T$ is the eigenvector related to $\lambda_{51} = 0$ for $[J(\mathbf{e}_5)]^T$. In addition to above, the calculation shows that,

$$\tilde{\tilde{\varphi}}^T \left[\frac{df}{dr}(\mathbf{e}_5, r^*)\right] = 0,$$

$$\tilde{\tilde{\varphi}}^T [\frac{d}{dX} f_r(\mathbf{e}_5, r^*)\tilde{\tilde{V}}] = 1 \neq 0,$$

$$\tilde{\tilde{\varphi}}^T [\frac{d^2}{dX^2} f(\mathbf{e}_5, r^*)(\tilde{\tilde{V}}, \tilde{\tilde{V}})] = 2\left[-\frac{a_1\tilde{\tilde{y}}(1 - c\tilde{\tilde{y}})}{aK} + \frac{a_1\tilde{\tilde{y}}(1 - c\tilde{\tilde{y}})^2}{a^2} - \frac{a_1\left(1 - 2c\tilde{\tilde{y}}\right)}{a}\rho_1\right] \neq 0.$$

Thus, the TCB occurs and the proof is complete. □

Theorem 5.5. *The z free EP,* $\mathbf{e}_6 = (\check{\check{x}}, \check{\check{y}}, 0)$, *is asymptotically stable locally if the following requirements occur,*

$$\frac{a_1(1 - c\check{\check{y}})^2\check{\check{y}}}{\left[a + \check{\check{x}}\left(1 - c\check{\check{y}}\right)\right]^2} < \frac{r}{K}, \tag{18}$$

$$\check{\check{y}} < \frac{1}{2c}, \tag{19}$$

$$\frac{b_2\check{\check{y}}}{b + \check{\check{y}}} + A_1 < d_2. \tag{20}$$

However, there is a TCB a round $\mathbf{e}_6$ *when* $d_2 = A_1 + \frac{b_2\check{\check{y}}}{b + \check{\check{y}}}$ $(\equiv d_2^*)$ *provided that,*

$$\frac{bb_2}{(b + \check{\check{y}})^2}\omega_2 - A_2 \neq 0. \tag{21}$$

Proof. Straightforward calculation gives that the VM of the system (2) at $\mathbf{e}_6$ is written as,

$$J(\mathbf{e}_6) = \begin{bmatrix} -\frac{r\check{\check{x}}}{K} + \frac{a_1(1 - c\check{\check{y}})^2\check{\check{x}}\check{\check{y}}}{[a + \check{\check{x}}(1 - c\check{\check{y}})]^2} & \frac{\check{\check{x}}\left(aa_1(1 - 2c\check{\check{y}}) + a\check{\check{x}}(1 - c\check{\check{y}})^2\right)}{[a + \check{\check{x}}(1 - c\check{\check{y}})]^2} & 0 \\ \frac{aa_2\check{\check{y}}(1 - c\check{\check{y}})}{[a + \check{\check{x}}(1 - c\check{\check{y}})]^2} & -\frac{s\check{\check{y}}}{L} - \frac{aa_2c\check{\check{x}}\check{\check{y}}}{[a + \check{\check{x}}(1 - c\check{\check{y}})]^2} & -\frac{b_1\check{\check{y}}}{b + \check{\check{y}}} \\ 0 & 0 & \frac{b_2\check{\check{y}}}{b + \check{\check{y}}} + A_1 - d_2 \end{bmatrix} = \left[\check{\check{b}}_{ij}\right].$$

Therefore, the eigenvalues are computed by,

$$\lambda_{61} = \frac{T_6}{2} - \frac{1}{2}\sqrt{T_6{}^2 - 4D_6}, \lambda_{62} = \frac{T_6}{2} + \frac{1}{2}\sqrt{T_6{}^2 - 4D_6}, \lambda_{63} = \frac{b_2\check{\check{y}}}{b + \check{\check{y}}} + A_1 - d_2,$$

where,

$$T_6 = \check{\check{x}} \left[-\frac{r}{K} + \frac{a_1(1-c\check{\check{y}})^2\check{\check{y}}}{[a+\check{\check{x}}(1-c\check{\check{y}})]^2} \right] + \check{\check{y}} \left[-\frac{s}{L} - \frac{aa_2c\check{\check{x}}}{[a+\check{\check{x}}(1-c\check{\check{y}})]^2} \right],$$

$$D_6 = \check{\check{x}}\check{\check{y}} \left[-\frac{r}{K} + \frac{a_1(1-c\check{\check{y}})^2\check{\check{y}}}{[a+\check{\check{x}}(1-c\check{\check{y}})]^2} \right] \left[-\frac{s}{L} - \frac{aa_2c\check{\check{x}}}{[a+\check{\check{x}}(1-c\check{\check{y}})]^2} \right] + \left[\frac{\check{\check{x}}\left(aa_1(1-2c\check{\check{y}}) + a\check{\check{x}}(1-c\check{\check{y}})^2\right)}{[a+\check{\check{x}}(1-c\check{\check{y}})]^2} \right] \left[\frac{aa_2\check{\check{y}}(1-c\check{\check{y}})}{[a+\check{\check{x}}(1-c\check{\check{y}})]^2} \right].$$

Verifying that is simple all the above eigenvalues have negative real parts under the conditions (18)-(20). Hence the solutions of system (2) approach asymptotically to $\mathbf{e}_6$ locally.

Now, when $d_2 = d_2^*$, further computation leads to that $\lambda_{63} = 0$ with corresponding eigenvector $\check{\check{V}} = (\omega_1, \omega_2, 1)^T$, where $\omega_1 = \dfrac{\check{\check{b}}_{12}\check{\check{b}}_{23}}{\check{\check{b}}_{11}\check{\check{b}}_{22} - \check{\check{b}}_{12}\check{\check{b}}_{21}} < 0$ and $\omega_2 = -\dfrac{\check{\check{b}}_{11}\check{\check{b}}_{23}}{\check{\check{b}}_{11}\check{\check{b}}_{22} - \check{\check{b}}_{12}\check{\check{b}}_{21}} > 0$. However, the eigenvector corresponding the zero eigenvalues of $[J(\mathbf{e}_6)]^T$ is given by $\check{\check{\varphi}} = (0,0,1)^T$. In addition to the above we obtain that,

$$\check{\check{\varphi}}^T \left[\frac{df}{dd_2}(\mathbf{e}_6, d_2^*) \right] = 0,$$

$$\check{\check{\varphi}}^T \left[\frac{d}{dX} f_{d_2}(\mathbf{e}_6, d_2^*)\check{\check{V}} \right] = -1 \neq 0,$$

$$\check{\check{\varphi}}^T \left[\frac{d^2}{dX^2} f(\mathbf{e}_6, d_2^*)(\check{\check{V}}, \check{\check{V}}) \right] = 2 \left[\frac{bb_2}{(b+\check{\check{y}})^2}\omega_2 - A_2 \right] \neq 0.$$

Thus, TCB a round $\mathbf{e}_6$ takes place under the condition (21). □

Theorem 5.6. *The survival point, $\mathbf{e}_7 = (x^*, y^*, z^*)$, is asymptotically stable locally if the following requirements occur,*

$$\frac{a_1(1-cy^*)^2 y^*}{[a + x^*(1-cy^*)]^2} < \frac{r}{K}, \tag{22}$$

$$y^* < \frac{1}{2c}, \tag{23}$$

$$\frac{b_1 z^*}{(b+y^*)^2} < \frac{s}{L} + \frac{aa_2cx^*}{[a+x^*(1-cy^*)]^2}. \tag{24}$$

While saddle-node bifurcation (SNB) takes place a round $\mathbf{e}_7$ when $A_2 = \dfrac{m_{11}m_{23}m_{32}}{(m_{12}m_{21} - m_{11}m_{22})z^*} (\equiv A_2^*)$ provided that,

$$\eta_1 {c_{11}}^* + \eta_2 {c_{21}}^* + {c_{31}}^* \neq 0. \tag{25}$$

Proof. Straightforward calculation gives that the VM of the system (2) at $\mathbf{e}_7$ is written as,

$$J(\mathbf{e}_7) = \begin{bmatrix} -\frac{rx^*}{K} + \frac{a_1(1-cy^*)^2x^*y^*}{[a+x^*(1-cy^*)]^2} & -\frac{a_1x^*\left[a(1-2cy^*)+x^*(1-cy^*)^2\right]}{[a+x^*(1-cy^*)]^2} & 0 \\ \frac{aa_2(1-cy^*)y^*}{[a+x^*(1-cy^*)]^2} & -\frac{sy^*}{L} - \frac{aa_2cx^*y^*}{[a+x^*(1-cy^*)]^2} + \frac{b_1y^*z^*}{(b+y^*)^2} & -\frac{b_1y^*}{(b+y^*)} \\ 0 & \frac{bb_2z^*}{(b+y^*)^2} & -A_2z^* \end{bmatrix},$$
$$= [m_{ij}].$$

Then, the eigenvalues can be computed from the equation,

$$\lambda^3 + G_1\lambda^2 + G_2\lambda + G_3 = 0,$$

where:

$$\begin{aligned} G_1 &= -(m_{11}+m_{22}+m_{33}), \\ G_2 &= m_{11}m_{22} - m_{12}m_{21} + m_{11}m_{33} + m_{22}m_{33} - m_{23}m_{32}, \\ G_3 &= -(m_{11}m_{22}m_{33} - m_{11}m_{23}m_{32} - m_{12}m_{21}m_{33}). \end{aligned}$$

It is observed that the conditions (22)-(24) ensure that $G_1 > 0$; $G_3 > 0$ and $\Delta = G_1G_2 - G_3 > 0$. Thus all roots of the above third-order equation have negative real parts due to the Routh-Hurwitz method. So, $\mathbf{e}_7$ is locally asymptotic stable.

Now simple calculation shows that $G_3 = 0$, when $A_2 = A_2^*$, then $J(\mathbf{e}_7)$ has $\lambda^* = 0$ with the other two negative real parts eigenvalues. Moreover, it has obtained that the eigenvectors related with $\lambda^* = 0$ for $J(\mathbf{e}_7)$ and $[J(\mathbf{e}_7)]^T$ are given by $V^* = (\tau_1, \tau_2, 1)^T$ and $\varphi^* = (\eta_1, \eta_2, 1)^T$ respectively, where,

$$\begin{aligned} \tau_1 &= \frac{m_{12}m_{23}}{m_{11}m_{22}-m_{12}m_{21}} < 0, \quad & \tau_2 &= \frac{-m_{11}m_{23}}{m_{11}m_{22}-m_{12}m_{21}} > 0, \\ \eta_1 &= \frac{m_{21}m_{32}}{m_{11}m_{22}-m_{12}m_{21}} < 0, \quad & \eta_2 &= \frac{-m_{11}m_{32}}{m_{11}m_{22}-m_{12}m_{21}} < 0. \end{aligned}$$

In addition, it has been obtained that,

$$\varphi^{*T}\left[\frac{df}{dA_2}(\mathbf{e}_7, A_2^*)\right] = -{z^*}^2 \neq 0,$$
$$\varphi^{*T}[\frac{d^2}{dX^2}f(\mathbf{e}_7, A_2^*)(V^*, V^*)] = \eta_1{c_{11}}^* + \eta_2{c_{21}}^* + {c_{31}}^*,$$

where,

$$\begin{aligned} {c_{11}}^* = 2&\left[-\frac{r}{K} + \frac{aa_1y^*(1-cy^*)^2}{(a+x^*(1-cy^*))^3}\right]{\tau_1}^2 + \frac{2aa_1cx^*(a+x^*)}{(a+x^*(1-cy^*))^3}{\tau_2}^2 \\ &- \frac{2aa_1(a(1-2cy^*)+x^*(1-cy^*))}{(a+x^*(1-cy^*))^3}\tau_1\tau_2, \end{aligned}$$

$$c_{21}{}^{*} = -\frac{2aa_1 y^*(1-cy^*)^2}{(a+x^*(1-cy^*))^3}\tau_1{}^2 - \frac{2bb_1}{(b+y)^2}\tau_2 + \frac{2aa_1(a(1-2cy^*)+x^*(1-cy^*))}{(a+x^*(1-cy^*))^3}\tau_1\tau_2$$
$$-2\left[\frac{s}{L} + \frac{aa_2cx^*(a+x^*)}{(a+x^*(1-cy^*))^3} - \frac{bb_1z^*}{(b+y^*)^3}\right]\tau_2{}^2,$$
$$c_{31}{}^{*} = \frac{-2bb_2z^*}{(b+y^*)^3}\tau_2{}^2 + \frac{2bb_2}{(b+y^*)^2}\tau_2 - 2A_2^*.$$

Therefore, $\varphi^{*T}[D^2 f(E_7, A_2^*)(V^*, V^*)] \neq 0$ under the condition (25), and hence SNB occurs. □

6 Worldwide Stability

The Lyapunov approach for stability is utilized in this part to examine the worldwide stability of EPs of the system (2) whenever possible. The worldwide stability requirements of these points are established in the next theorems. Since it is commonly known that locally unstable points cannot be worldwide stable, we restrict our analysis in the following theorems to locally stable points.

Theorem 6.1. *The EP,* $\mathbf{e}_1 = (\breve{x}, 0, 0) = (K, 0, 0)$, *is a worldwide asymptotically stable* (*WAS*) *provided that,*

$$s + \frac{a_2\breve{x}}{a} < d_1. \tag{26}$$

Proof. Consider the function $L_1 = \frac{a_2}{a_1}\left(x - \breve{x} - \breve{x}\ln\frac{x}{\breve{x}}\right) + y + \frac{b_1}{b_2}z$, then we have,

$$\frac{dL_1}{dt} = -\frac{a_2}{a_1}\frac{r}{K}(x-\breve{x})^2 + \breve{x}\frac{a_2(1-cy)y}{a+x(1-cy)} + (s-d_1)y - \frac{s}{L}y^2 + \frac{b_1}{b_2}(A_1-d_2)z - \frac{b_1}{b_2}A_2z^2.$$

By doing some algebraic steps, it is obtained that,

$$\frac{dL_1}{dt} \leq -\frac{a_2}{a_1}\frac{r}{K}(x-\breve{x})^2 + \left[\frac{a_2\breve{x}}{a} + s - d_1\right]y + \frac{b_1}{b_2}[A_1 - d_2]z.$$

Therefore, $\frac{dL_1}{dt} < 0$ under the conditions (26) and (11), hence $\mathbf{e}_1$ is a WAS. □

Theorem 6.2. *The EP,* $\mathbf{e}_2 = (0, \hat{y}, 0) = \left(0, \frac{(s-d_1)L}{s}, 0\right)$ *is WAS provided that,*

$$\frac{b_2\hat{y}}{b} + A_1 < d_2, \tag{27}$$
$$\frac{r}{a_1} < \frac{\hat{y}}{a+K}. \tag{28}$$

Proof. Let $L_2 = \frac{a_2}{a_1}x + \left(y - \hat{y} - \hat{y}\ln\frac{y}{\hat{y}}\right) + \frac{b_1}{b_2}z$ be a scalar positive definite function. Then we have,

$$\frac{dL_2}{dt} = \frac{a_2}{a_1}rx\left(1 - \frac{x}{K}\right) - \frac{s}{L}(y-\hat{y})^2 - \frac{a_2(1-cy)\hat{y}x}{a+x(1-cy)} + \frac{b_1\hat{y}z}{b+y} + \frac{b_1}{b_2}A_1z - \frac{b_1}{b_2}A_2z^2 - \frac{b_1}{b_2}d_2z.$$

By doing some algebraic manipulations, it results that,

$$\frac{dL_2}{dt} < -\left[\frac{\hat{y}}{a+K} - \frac{r}{a_1}\right] a_2 x - \frac{s}{L}(y-\hat{y})^2 - b_1 z\left[\frac{d_2}{b_2} - \frac{\hat{y}}{b} - \frac{A_1}{b_2}\right].$$

Therefore, $\frac{dL_2}{dt} < 0$ provided the conditions (27) and (28) hold. Hence $\mathbf{e}_2$ is a WAS. □

Theorem 6.3. *The y free EP,* $\mathbf{e}_4 = (\breve{x}, 0, \overline{z}) = \left(K, 0, \frac{A_1 - d_2}{A_2}\right)$ *is a WAS if the following condition holds,*

$$s + \frac{a_2 \breve{x}}{a} < d_1 + \frac{b_1 \overline{z}}{b}. \tag{29}$$

Proof. Consider the function $L_3 = \frac{a_2}{a_1}\left(x - \breve{x} - \breve{x}\ln\frac{x}{\breve{x}}\right) + y + \frac{b_1}{b_2}\left(z - \overline{z} - \overline{z}\ln\frac{z}{\overline{z}}\right)$, then we have,

$$\frac{dL_3}{dt} = -\frac{a_2}{a_1}\frac{r}{K}(x-\breve{x})^2 + \breve{x}\frac{a_2(1-cy)y}{a+x(1-cy)} + s\left(1-\frac{y}{L}\right)y - d_1 y - \frac{b_1\overline{z}y}{b+y} - \frac{b_1}{b_2}A_2(z-\overline{z})^2.$$

Moreover, it is determined that,

$$\frac{dL_3}{dt} \leq -\frac{a_2}{a_1}\frac{r}{K}(x-\breve{x})^2 + \left[\frac{a_2\breve{x}}{a} + s - d_1 - \frac{b_1\overline{z}}{b}\right] y - \frac{b_1}{b_2}A_2(z-\overline{z})^2.$$

Hence, $\frac{dL_3}{dt} < 0$ if the condition (29) holds. Therefore, $\mathbf{e}_4$ is a WAS. □

Theorem 6.4. *The x free EP,* $\mathbf{e}_5 = (0, \tilde{\tilde{y}}, \tilde{\tilde{z}})$, *is a WAS when the following conditions hold,*

$$\frac{r}{a_1} < \frac{\tilde{\tilde{y}}}{a+K}, \tag{30}$$

$$\frac{b_1\tilde{\tilde{z}}}{b\left(b+\tilde{\tilde{y}}\right)} < \frac{s}{L}. \tag{31}$$

Proof. Let $L_4 = \frac{a_2}{a_1}x + y - \tilde{\tilde{y}} - \tilde{\tilde{y}}\ln\frac{y}{\tilde{\tilde{y}}} + \frac{b_1(b+\tilde{\tilde{y}})}{bb_2}\left(z - \tilde{\tilde{z}} - \tilde{\tilde{z}}\ln\frac{z}{\tilde{\tilde{z}}}\right)$, then we have,

$$\frac{dL_4}{dt} = r\frac{a_2}{a_1}x\left(1-\frac{x}{K}\right) - \left[\frac{s}{L} - \frac{b_1\tilde{\tilde{z}}}{(b+y)\left(b+\tilde{\tilde{y}}\right)}\right]\left(y-\tilde{\tilde{y}}\right)^2 - \frac{a_2(1-cy)\tilde{\tilde{y}}x}{a+x(1-cy)} - \frac{b_1(b+\tilde{\tilde{y}})}{bb_2}A_2(z-\tilde{\tilde{z}})^2.$$

Furthermore, it is obtained that,

$$\frac{dL_4}{dt} < -\left[\frac{\tilde{\tilde{y}}}{a+K} - \frac{r}{a_1}\right]a_2 x - \left[\frac{s}{L} - \frac{b_1\tilde{\tilde{z}}}{b\left(b+\tilde{\tilde{y}}\right)}\right]\left(y-\tilde{\tilde{y}}\right)^2 - \frac{b_1\left(b+\tilde{\tilde{y}}\right)}{b_2 b}A_2(z-\tilde{\tilde{z}})^2.$$

Therefore, due to the given conditions, $\frac{dL_4}{dt} < 0$. Thus, $\mathbf{e}_5$ is a WAS. □

Theorem 6.5. *The z free EP,* $\mathbf{e}_6 = (\breve{\breve{x}}, \breve{\breve{y}}, 0)$*, is a WAS under the conditions,*

$$
\begin{aligned}
{q_{12}}^2 &< 4q_{11}q_{22}, & (32)\\
\frac{r}{K} &> \frac{a_1\breve{\breve{y}}\,(1-cy)\left(1-c\breve{\breve{y}}\right)}{R\breve{\breve{R}}}, & (33)\\
d_2 &> \frac{b_2}{b_1}\breve{\breve{y}} + A_1, & (34)
\end{aligned}
$$

where $R = a + x(1-cy)$ *and* $\breve{\breve{R}} = a + \breve{\breve{x}}(1-c\breve{\breve{y}})$*, while* q_{11}, q_{12} *and* q_{22} *are stated in the proof.*

Proof. Define the function $L_5 = x - \breve{\breve{x}} - \breve{\breve{x}}\ln\frac{x}{\breve{\breve{x}}} + y - \breve{\breve{y}} - \breve{\breve{y}}\ln\frac{y}{\breve{\breve{y}}} + \frac{b_1}{b_2}z$, then by doing some algebraic steps yield that,

$$
\frac{dL_5}{dt} \le -q_{11}(x-\breve{\breve{x}})^2 - q_{22}(y-\breve{\breve{y}})^2 + q_{12}\left(x-\breve{\breve{x}}\right)\left(y-\breve{\breve{y}}\right) - \left[\frac{b_1(d_2-A_1)}{b_2} - \breve{\breve{y}}\right]z,
$$

where,

$$
\begin{aligned}
q_{11} &= \frac{r}{K} - \frac{a_1\breve{\breve{y}}\,(1-cy)\left(1-c\breve{\breve{y}}\right)}{R\breve{\breve{R}}},\\
q_{22} &= \frac{s}{L} + \frac{aa_2cx}{R\breve{\breve{R}}},\\
q_{12} &= \frac{aa_2\left(1-c\breve{\breve{y}}\right) + a_1c^2\breve{\breve{x}}\breve{\breve{y}}y + a_1c\left(y+\breve{\breve{y}}\right)\left(a+\breve{\breve{x}}\right) - a_1\left(a+\breve{\breve{x}}\right)}{R\breve{\breve{R}}}.
\end{aligned}
$$

Now by using the given conditions we obtains that,

$$
\frac{dL_5}{dt} \le -\left[\sqrt{q_{11}}\left(x-\breve{\breve{x}}\right) - \sqrt{q_{22}}\left(y-\breve{\breve{y}}\right)\right]^2 - \left[\frac{b_1(d_2-A_1)}{b_2} - \breve{\breve{y}}\right]z.
$$

Therefore, $\frac{dL_5}{dt} < 0$, and hence $\mathbf{e}_6$ is a WAS. □

Theorem 6.6. *The survival point,* $\mathbf{e}_7 = (x^*, y^*, z^*)$ *is a WAS under the conditions,*

$$
\begin{aligned}
\frac{a_1y^*}{aR^*}(1-cy^*) &< \frac{r}{K}, & (35)\\
\frac{z^*}{B^*} &< \frac{s}{L}, & (36)\\
{p_{12}}^2 &< 2p_{11}p_{22}, & (37)\\
{p_{23}}^2 &< 2p_{22}p_{33}, & (38)
\end{aligned}
$$

where $B = b + y$; $B^* = b + y^*$, $R^* = a + x^*(1-cy^*)$*, while* p_{ij} *are given below.*

Proof. Let $L_6 = x - x^* - x^*\ln\frac{x}{x^*} + y - y^* - y^*\ln\frac{y}{y^*} + z - z^* - z^*\ln\frac{z}{z^*}$. Then using some mathematical steps, it is observed,

$$
\frac{dL_6}{dt} \le -p_{11}(x-x^*)^2 + p_{12}(x-x^*)(y-y^*) - p_{22}(y-y^*)^2 + p_{23}(y-y^*)(z-z^*) - p_{33}(z-z^*)^2,
$$

where,

$$\begin{aligned}
p_{11} &= \frac{r}{K} - \frac{a_1 y^*}{RR^*}(1 - cy^*)(1 - cy),\\
p_{22} &= \frac{s}{L} - \frac{bz^*}{BB^*} + \frac{aa_1 cx}{RR^*},\\
p_{12} &= \frac{a_1}{RR^*}\left[c\,(y + y^*)\,(a + x^*) - \left(a + x^*\left(1 + c^2 y^* y\right)\right) + a(1 - cy^*)\right],\\
p_{23} &= \frac{b}{BB^*}\,(b_2 - B^*)\,,\\
p_{33} &= A_2.
\end{aligned}$$

Using the above conditions yield that,

$$\frac{dL_6}{dt} \leq -\left[\sqrt{p_{11}}\,(x - x^*) - \sqrt{\frac{p_{22}}{2}}\,(y - y^*)\right]^2 - \left[\sqrt{\frac{p_{22}}{2}}\,(y - y^*) - \sqrt{p_{33}}\,(z - z^*)\right]^2.$$

Therefore, $\frac{dL_6}{dt} < 0$, and hence $\mathbf{e}_7$ is a WAS. □

7 Numerical Simulation

Two goals are achieved in this part, the first of which is the validation of the analytical results. The second goal is to investigate the role of changing the system's parameters (2) on its asymptotic behavior. Now, beginning from three different initial locations as depicted in Figure (1), system (2) approaches asymptotically to a survival point that is provided by $\mathbf{e}_7 = (43.6,\ 10.61,\ 7.36)$,

$$\begin{array}{llllll}
r = 1, & s = 1, & K = 40, & L = 40, & a_1 = 1, & a = 10,\\
c = 0.1, & a_2 = 0.75, & b_1 = 1, & b = 10, & b_2 = 0.75, & \\
A_1 = 0.5, & A_2 = 0.1, & d_1 = 0.1, & d_2 = 0.15. & &
\end{array} \tag{39}$$

Obviously, Figure 1 shows clearly the existence of a WAS to a survival point as proved in Theorem 6.6.

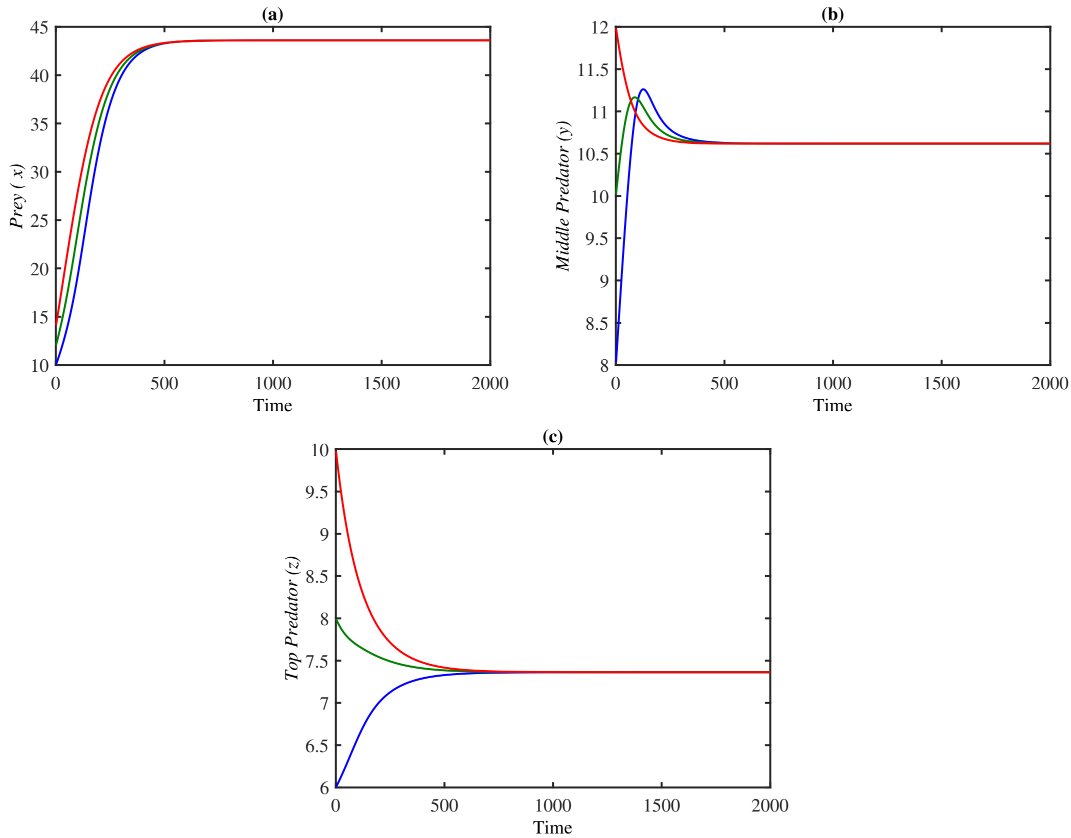

Figure 1: Time series for the trajectory of the system (2) using the set (39) that approaches asymptotically to $\mathbf{e}_7$.

As demonstrated in Figure 2, the trajectory of the system (2) now moves asymptotically to the y free EP, $\mathbf{e}_4 = (40, 0, 3.5)$ for the set (39) with $b \leq 1.5$. This supports the analytical findings of Theorem 6.3.

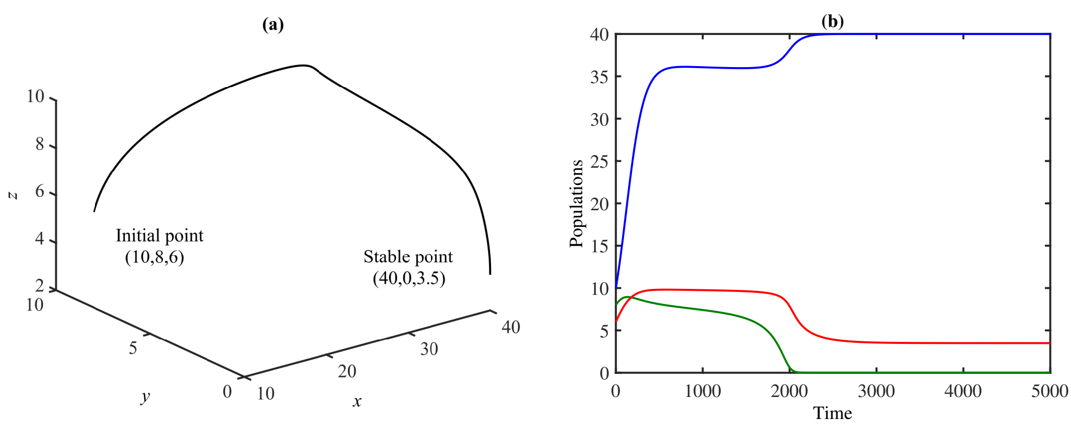

Figure 2: The trajectory of the system (2) using (39) with $b = 1.5$: (**a**) System (2) approaches asymptotically to $\mathbf{e}_4$. (**b**) Trajectories versus time in (a).

According to the typical figure provided by Figure 3, the trajectory of the system (2) goes asymptotically to the x free EP for the set (39) with $c \leq 0.02$. Once more, this supports the analytical finding presented in Theorem 6.4.

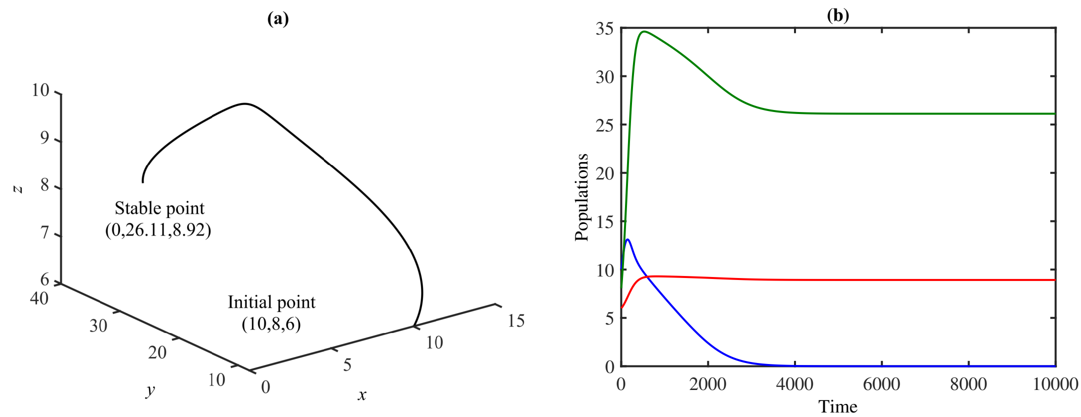

Figure 3: The trajectory of system (2) using (39) with $c = 0.02$: (**a**) System (2) approaches asymptotically to $\mathbf{e}_5 = (0, 26.11, 8.92)$. (**b**) Trajectories versus time in (a).

It is noted that the trajectory of system (2) returns to the y free EP as shown in Figure 2 for the data (39) with $b_1 \geq 4.3$.

The trajectory of system (2) approaches asymptotically to y free EP for the data (39) with $A_1 \geq 1.79$ or $A_2 \leq 0.02$, as seen in the typical figures illustrated by Figures 4 and 5, respectively.

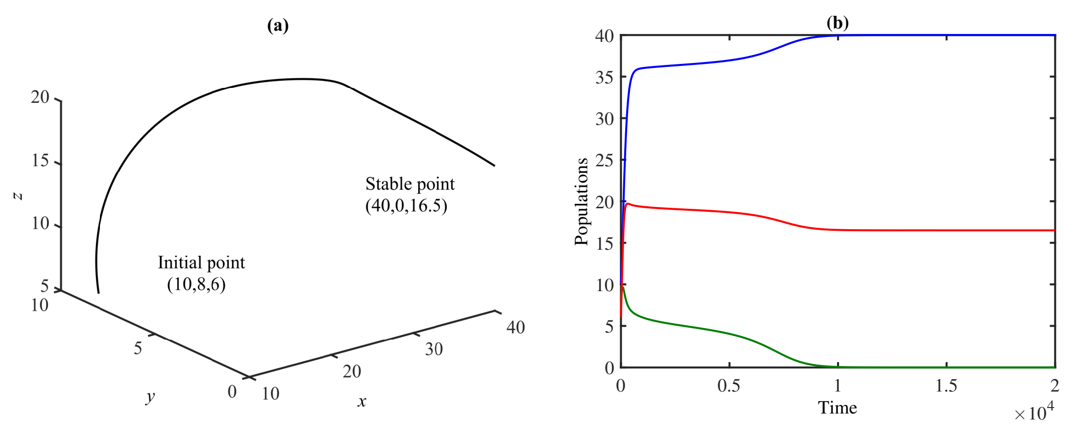

Figure 4: The trajectory of the system (2) using (39) with $A_1 = 1.8$: (**a**) System (2) approaches asymptotically to $\mathbf{e}_4 = (40, 0, 16.5)$. (**b**) Trajectories versus time in (a).

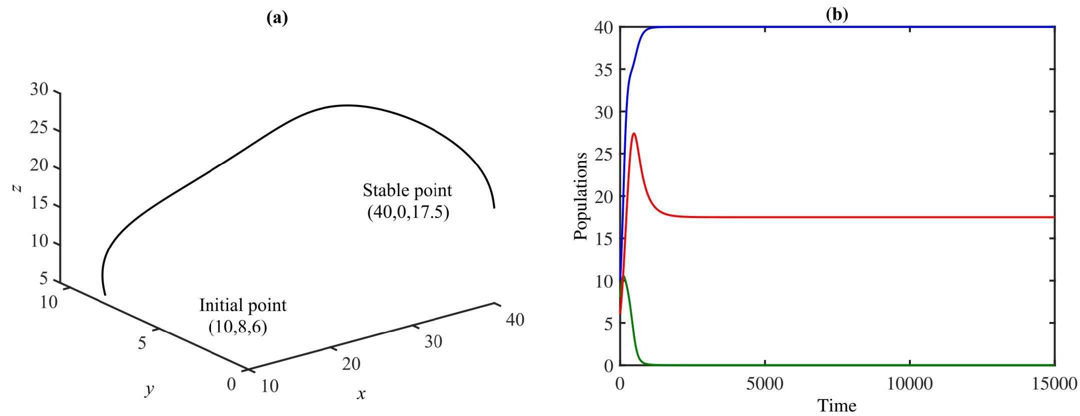

Figure 5: The trajectory of the system (2) using (39) with $A_2 = 0.02$: (**a**) System (2) approaches asymptotically to $\mathbf{e}_4 = (40, 0, 17.5)$. (**b**) Trajectories versus time in (a).

The trajectory of the system (2) now moves asymptotically to the z free equilibrium point as shown in the typical figure provided by Figure 6 for the set (39) with $d_2 \geq 0.9$. This supports the analytical findings of Theorem 6.5.

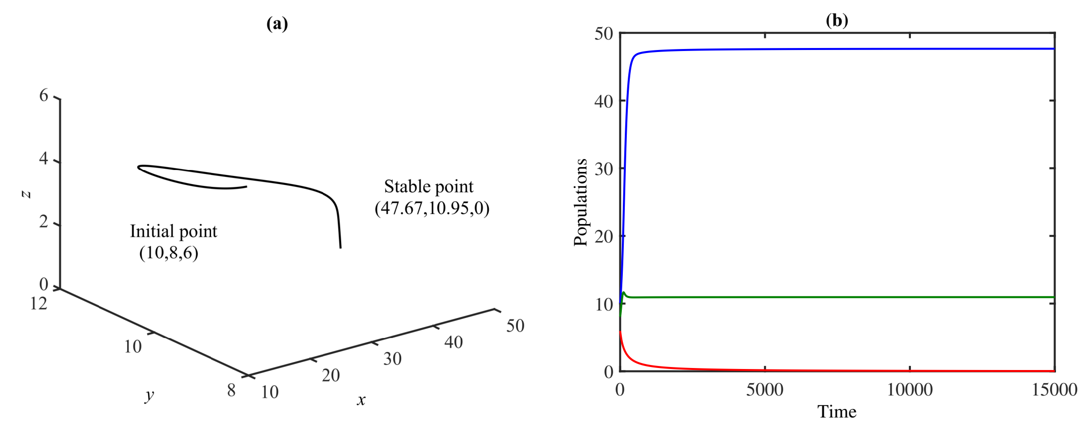

Figure 6: The trajectory of the system (2) using (39) with $d_2 = 0.9$: (**a**) System (2) approaches asymptotically to $\mathbf{e}_6 = (47.67, 10.95, 0)$. (**b**) Trajectories versus time in (a).

The position of the survival point is also seen to shift quantitatively when one of the parameters r, s, K, L, a, a_1, a_2, and d_1 is changed using the hypothetical set (39) in consideration. This occurred because of the given data's inability to satisfy the bifurcation conditions, and as a result, the bifurcation may occur for another set of parameters.

On the other hand, the trajectory of the system (2) converges asymptotically to the y free EP for the data (39) with $s = 0.1$ and $d_1 = 0.8$, indicating that condition (10) holds but condition (11) does not. The system (2) does, however, asymptotically approach the $\mathbf{e}_1$ after the parameter A_1 lowers so that condition (11) applies, as seen in Figure 7.

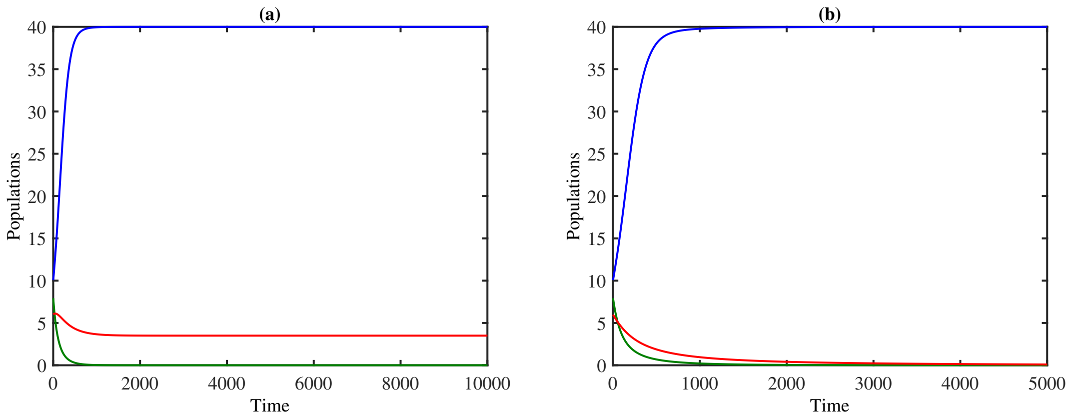

Figure 7: (**a**) The time series of the system (2), using (39) with $s = 0.1$ and $d_1 = 0.8$, which approaches asymptotically to $\mathbf{e}_4 = (40, 0, 3.5)$. (**b**) The time series of the system (2), using (39) with $s = 0.1$, $d_1 = 0.8$, and $A_1 = 0.12$, which approaches asymptotically to $\mathbf{e}_1 = (40, 0, 0)$.

Finally, the trajectory of the system (2) approaches asymptotically to the x free EP for the data (39) with $A_1 = 0.1$ and $c = 0.01$, condition (12) holds whereas the condition given by (13) does not. The system (2) does, however, asymptotically approach the $\mathbf{e}_2$ point after the parameter d_2 grows such that condition (13) applies, as seen in Figure 8.

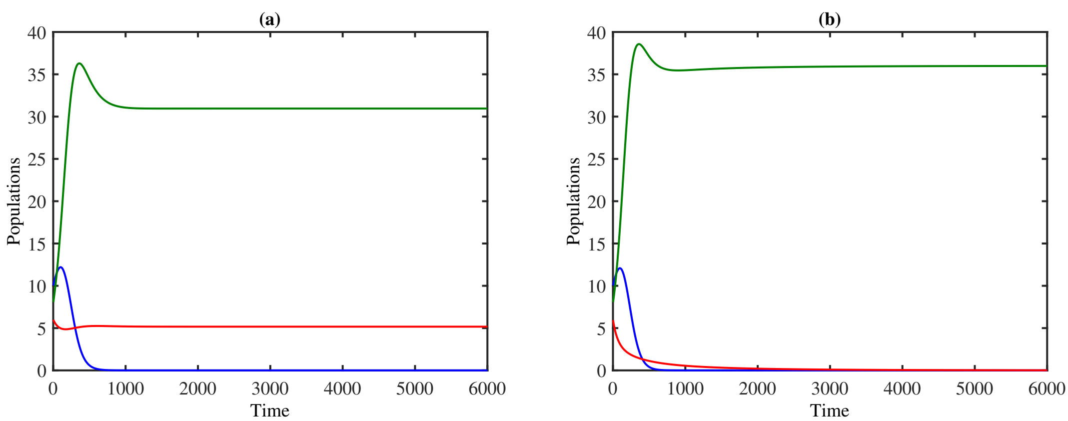

Figure 8: (**a**) The time series of the trajectory of the system (2), using (39) with $A_1 = 0.1$ and $c = 0.01$, which approaches asymptotically to $\mathbf{e}_5 = (0, 30.95, 5.16)$. (**b**) The time series of the trajectory of the system (2), using data (34) with $A_1 = 0.1$, $c = 0.01$, and $d_2 = 0.75$, which approaches asymptotically to $\mathbf{e}_2 = (0, 36, 0)$.

8 Discussion and Conclusions

An ecological model with a food chain that includes refugees at the first level and is proportional to the presence of mid-predators with the inclusion of additional food sources in the second and third levels is presented and examined in this research. There are shown to be all potential EPs. The system's local and worldwide stability are both investigated. All of the local bifurcation conditions that ensure bifurcation will occur near the EPs have been defined. The system is ultimately numerically simulated to confirm what we discovered and understand the effects of altering parameter values on the system's asymptotic behavior. The system is seen to have rich dynamics and to be responsive to changing parameter values. In fact, it has been found that raising the amount of refugees in the population has a stabilizing effect on the system's dynamic behavior. Expanding the top predator's alternative food sources causes the mid-predator to go extinct and destabilizes the ecosystem as a whole. However, it was found that additional food has a negative effect on predator biomass, which means additional food does not always enhance the growth of predators [24]. Mondal et al. [23] have observed that the predator, prey, and subsidy can always exist at a nonzero subsidy input rate, while at a high subsidy input rate, the prey population cannot persist and the predator population is growing hugely due to the availability of food sources.

Acknowledgement We would like to thank anonymous reviewers for all their constructive comments and recommendations for the betterment of this paper.

Conflicts of Interest The authors say they have no competing interests.

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